Every Gevrey-α vector field with nilpotent linear part admits a Gevrey- $(1+\alpha)$ normal form

Freek Verstringe
joint work with P. Bonckaert
Hasselt University, Belgium

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- Some general notions on normal forms.
- A reminder of $\mathrm{sl}(2,\mathbb{C})$-representations and the construction of some special $\mathrm{sl}(2,\mathbb{C})$-representations.
- Combine both ideas to obtain our main result.

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$$X = N + f, \;\; X' = N + g$$

$$\begin{aligned} &\Phi_*(X) = N + g \\ \Leftrightarrow\; & X \circ \Phi^{-1} = D\Phi^{-1}.X' \\ \Leftrightarrow\; & (N + f) \circ (I + u) = D(I + u).(N + g) \\ \Leftrightarrow\; & g + [u, N] = f(I + u) - Du.g \end{aligned}$$

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  is not surjective.
- Complementary space: $\mathcal{C}_\delta \oplus \mathrm{Im}(d_{0,\delta}) = \mathcal{V}_\delta$ is needed. How to choose this space?

- Suppose $N = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \ldots + \lambda_n x_n \frac{\partial}{\partial x_n}$

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- Convergence if all eigenvalues < 0, or > 0 or satisfy Brjuno condition.

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- Suppose $\forall k, j$ we have $(\langle \lambda, k\rangle - \lambda_j) \neq 0$.
- Suppose there is at least one positive and negative eigenvalue.
- This leads in a lot of cases to the divergence of the normal form procedure (even if the eigenvalues satisfy some diophantine condition).

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- A lot of formal results exist involving $\mathrm{sl}(2,\mathbb{C})$ representations.
- Recent framework (explained below) by Lombardi-Stolovitch lead to Gevrey-1 normal form in dimensions 2 and 3.
- We combine the recent framework with representation theory of $\mathrm{sl}(2,\mathbb{C})$ to generalize this result to any dimension.

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- $\mathcal{C}_\delta = \ker(d_0^*)$.

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The complementary subspace

- Inner product on $\mathcal{P}_\delta$.
- $$\left\langle \sum_{|\alpha|=\delta} a_\alpha x^\alpha, \sum_{|\beta|=\delta} b_\beta x^\beta \right\rangle = \sum_{|\alpha|=\delta} a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}.$$
- This induces an inner product on the space $\mathcal{V}_{\delta-1}$ of vector fields of degree $\delta - 1$ as follows:

$$\left\langle \sum_{k=1}^n V_k \frac{\partial}{\partial x_k}, \sum_{k=1}^n W_k \frac{\partial}{\partial x_k} \right\rangle = \sum_{k=1}^n \langle V_k, W_k \rangle_\delta ,$$

Where the V_k, W_k are elements of $\mathcal{P}_\delta$.

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- The adjoint is completely determined by

$$\langle d_0^*(V), W \rangle = \langle V, d_0(W) \rangle \,.$$

Adjoints(2)

To be very precise

$$\left(x_i \frac{\partial}{\partial x_{i+1}}\right)^* = \left(x_{i+1} \frac{\partial}{\partial x_i}\right),$$

$$d_0^*(V) = \begin{pmatrix} N^* - \left(\frac{\partial N_1}{\partial x_1}\right)^* & -\left(\frac{\partial N_2}{\partial x_1}\right)^* & \cdots & -\left(\frac{\partial N_n}{\partial x_1}\right)^* \\ -\left(\frac{\partial N_1}{\partial x_2}\right)^* & N^* - \left(\frac{\partial N_2}{\partial x_2}\right)^* & \cdots & -\left(\frac{\partial N_n}{\partial x_2}\right)^* \\ \vdots & & & \vdots \\ -\left(\frac{\partial N_1}{\partial x_n}\right)^* & \cdots & -\left(\frac{\partial N_{n-1}}{\partial x_n}\right)^* & N^* - \left(\frac{\partial N_n}{\partial x_n}\right)^* \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ \vdots \\ V_n \end{pmatrix}.$$

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  - $\mathcal{V}_\delta = \mathrm{Im}(\Box_\delta) \oplus \mathrm{Ker}(\Box_\delta) = \mathrm{Im}(d_0) \oplus \mathrm{Ker}(d_0^*)$
- What about the convergence/divergence of the transformation $I + u$ and normal form $X' = N + g$?

Theorems by Iooss-Lombardi and Lombardi-Stolovitch

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Suppose that $X = N + f$ is formally linearizable and N satisfies a diophantine condition, then X is also analytically linearizable.

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Suppose that $X = N + f$ has a formal normal form $X' = N + g$ by means of the procedure explained in this section, and suppose that N satisfies a Siegel condition of order τ, then X' and U are formal power series of type Gevrey-$(1+\tau)$.

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Theorem

Suppose that $X = N + f$ is formally linearizable and N satisfies a Siegel condition, then there exists an optimal δ to stop the normal form procedure. The transformation $id + u = id + u_2 + u_3 + \ldots + u_\delta$ transforms the vector field into $X' = N + g_2 + \ldots + g_\delta + R$ and R is exponentially small. (Technical to state precisely)

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  η_δ depends on a_i, $0 \leq i \leq \delta$.
- These conditions are automatically satisfied if $a_\delta \geq C_0 > 0$; C_0 independent of δ.

Some definitions concerning representations of Lie algebras

- A Lie algebra $(\mathfrak{g}, [\,,\,])$ is a vector space $\mathfrak{g}$ provided with a multiplication $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g} : (x, y) \mapsto [x, y]$ that satisfies the relations

$$[g_1, g_2] = -[g_2, g_1]$$
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- $gl_n(\mathbb{C})$, the group of $n \times n$-matrices is a Lie algebra with $[A, B] = AB - BA$.
- A linear mapping from a general Lie algebra $\mathfrak{g}$ to $gl_n(\mathbb{C})$ preserving the product structure is called a finite dimensional representation.

Representations of $\mathrm{sl}(2,\mathbb{C})$

- $\mathrm{sl}(2,\mathbb{C})$ is generated by the matrices

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- $\mathrm{sl}(2,\mathbb{C})$ is a simple Lie algebra, this is $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.
- Every finite dimensional representation of $\mathrm{sl}(2,\mathbb{C})$ can be written as a direct sum of irreducible representations.

A list of the irreducible representations of sl$(2,\mathbb{C})$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto N_n' := \begin{pmatrix} 0 & n-1 & 0 & 0 & \dots & 0 \\ 0 & 0 & n-2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto M_n' := \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & n-2 & 0 & 0 \\ 0 & 0 & \dots & 0 & n-1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_n' := \begin{pmatrix} n-1 & 0 & 0 & 0 & \dots & 0 \\ 0 & n-3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -n+3 & 0 \\ 0 & 0 & \dots & 0 & 0 & -n+1 \end{pmatrix}$$

Examples of representations of $\mathrm{sl}(2,\mathbb{C})$

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto y\frac{\partial}{\partial x}$$
$$\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto x\frac{\partial}{\partial y}$$
$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto [y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}]$$

Acting on the space $\mathcal{P}_\delta$ of polynomials in x, y of homogeneous degree δ.

Examples of representations of $\mathrm{sl}(2,\mathbb{C})$

- Find real values of α_i for which the following is a representation

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto N_n = \alpha_1 x_2 \frac{\partial}{\partial x_1} + \alpha_2 x_3 \frac{\partial}{\partial x_2} + \ldots + \alpha_n x_{n+1} \frac{\partial}{\partial x_n}$$

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- The $\mathrm{sl}(2,\mathbb{C})$-relations $[H,N] = 2N$, $[H,M] = -2M$, $[N,M] = H$ have to hold.
- This delivers $2n$ equations in n unknowns; luckily the equations generated by $[H,N] = 2N$ are basically the same as those generated by $[H,M] = -2M$.

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$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto N_n^* = M_n = \bar{\alpha}_1 x_1 \frac{\partial}{\partial x_2} + \bar{\alpha}_2 x_2 \frac{\partial}{\partial x_3} + \ldots + \bar{\alpha}_n x_n \frac{\partial}{\partial x_{n+1}}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_n = [N_n, M_n].$$

- The $\mathrm{sl}(2,\mathbb{C})$-relations $[H,N] = 2N$, $[H,M] = -2M$, $[N,M] = H$ have to hold.
- This delivers $2n$ equations in n unknowns; luckily the equations generated by $[H,N] = 2N$ are basically the same as those generated by $[H,M] = -2M$.
- This delivers $\alpha_i = \sqrt{i(n+1-i)}$.

Examples of representations of $\mathrm{sl}(2,\mathbb{C})$

Matrix of $d_0(\sum_{i=1}^{n+1} V_i \frac{\partial}{\partial x_i})$

$$\begin{pmatrix} N_n & -\alpha_1 I & 0 & 0\dots & 0 & 0 \\ 0 & N_n & -\alpha_2 I & 0\dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & N_n & -\alpha_n I \\ 0 & \dots & 0 & 0 & 0 & N_n \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ V_{n+1} \end{pmatrix}.$$

Matrix of $d_0^*(\sum_{i=1}^{n+1} V_i \frac{\partial}{\partial x_i})$ as remember that $M = N^*$)

$$\begin{pmatrix} M_n & 0 & 0 & 0\dots & 0 & \\ -\alpha_1 I & M_n & 0 & 0\dots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -\alpha_{n-1} I & M_n & 0 \\ 0 & \dots & 0 & 0 & -\alpha_n I & M_n \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ V_{n+1} \end{pmatrix}.$$

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- One computes that $[D, d_0] = 2d_0$ and $[D, d_0^*] = 2d_0^*$.
- Hence the triple d_0, d_0^*, D generates a representation of $\mathrm{sl}(2,\mathbb{C})$!

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- In general this is impossible.
- We prepare the linear part of the vector field to obtain a link with representation theory.

Proof of the main theorem(2)

- Prepare the linear part as $N_n = \sum_{i=1}^{n} \sqrt{i(n+1-i)}x_{i+1}\frac{\partial}{\partial x_i}$, using a theorem of Jordan.

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- What about multiple Jordan blocks?

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- Existence of result with exponentially small remainder.